Variable-Lift Re-Entry at Superorbital and Orbital Speeds

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Analytical solutions have been obtained for a type of variable-lift modulation applicable to a lifting body entering the atmosphere from superorbital and orbital speeds. The variable-lift modulation analyzed here results in nonoscillatory trajectories and achieves a smooth transition into a nominal glide phase such as constant flight path angle, constant altitude, constant dynamic pressure, or any other glide desired. Simple closed-form formulas were derived which permit quick estimation of 1) the peak deceleration and the altitude where it occurs, and 2) the required lift modulation necessary for any set of prescribed transition conditions. These approximate formulas show the explicit dependence of trajectory quantities in terms of re-entry conditions, vehicle aerodynamic characteristics, and parameters defining the lift modulation. It can be shown that the analytical solutions obtained in this report include the constant lift-drag solutions obtained by Lees, Hartwig, and Cohen in the orbital case and by Ting and Wang in the superorbital case. Furthermore, in the case of variable lift, this report deals with a continuous lift modulation program instead of a stepwise change of lift-drag considered by Lees. Hartwig, and Cohen.

Nomenclature

A = reference area, ft²
a_{ij} = coefficients
C_L = lift coefficient
C_D = drag coefficient
C_{D0} = zero-lift-drag coefficient
C_R = total force coefficient $(C_L^2 + C_D^2)^{1/2}$
F_1,F_2,F_3 = abbreviations, Eq. (20)
f_* = $\zeta X^{*n}/C_{LE}$
G = deceleration load, g's
g = gravitational acceleration, 32.2 ft/sec²
h = altitude, ft
k = lift-drag polar parameter, Eq. (9)
K_0,K_1,K_2 = constants, Eq. (47)
m = mass
n = lift parameter, Eq. (11)
q = dynamic pressure, psf
R = mean radius of earth, 2.092×10^7 ft
t = time, sec
V = velocity, fps
V_s = near-earth circular satellite velocity, 25,750 fps
W = weight, lb
X = transformed independent variable, Eq. (5)
Y = $(1/\beta R) \ln(X/X_E)$
Z = abbreviation for $\zeta[C_{LE}(C_{DE})^n]^{-1}$
β = atmospheric density decay parameter, 0.403×10^{-4} ft^{-1}
θ = flight-path angle, positive below local horizon
ρ = atmospheric density, slugs/ft³
ρ_r = reference density, 1.9×10^{-3} slugs/ft³
ζ = lift parameter, Eq. (11)

Subscripts

0 = for $\zeta = 0$
1 = at end of transition phase for transition into constant flight-path angle glide or constant altitude glide
2 = at end of transition phase for transition into constant dynamic pressure glide

Superscript

* = refers to peak deceleration

Presented at the IAS Annual Summer Meeting, Los Angeles, Calif., June 19–22, 1962; revision received March 14, 1963. This paper is a condensed version of Ref. 1.

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I. Introduction

An analysis using a continuous variable lift program is described in this paper. The analysis treats only the initial portion of the trajectory, i.e., before pull-out. The study was initiated by the need of vehicle design information for a manned re-entry with lift and drag modulation to achieve a smooth transition into a nominal glide phase from the initial plunge. As a re-entry vehicle plunges into the earth atmosphere at high speeds, it normally follows an oscillatory trajectory when the aerodynamic coefficients are held constant. Under this circumstance, it is not possible for the vehicle to get on any glide phase smoothly without varying C_L and C_D. Lift and drag modulation, therefore, becomes necessary. Two important features of lift modulation are studied in this paper: 1) the peak deceleration during re-entry with lift modulation and the altitude where it occurs; and 2) the requirements on the lift program to achieve a smooth transition. The lift program studied here is represented by the relation $C_L = C_{LE} - \zeta X^n$, where X is a transformed variable proportional to the atmospheric density, ζ and n are lift parameters, and C_{LE} is the lift coefficient at the point of entry. Such a lift program can be used to approximate most, if not all, lift programs. The drag coefficient is assumed to vary with X through a parabolic lift-drag polar.

Approximate solutions to the trajectory are obtained by a series approach and are applicable to both the orbital and superorbital re-entry. The results provide a reference framework for further parametric study of optimization using computers.

II. Equations of Motion and Lift Program

In studying the motion of hypervelocity vehicle entering the earth atmosphere, the vehicle can be treated as a point mass if stability is not to be investigated. In addition, the following assumptions can be made without introducing appreciable errors. These assumptions are small flight-path angle such that $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, nonrotating earth, small variation of altitude relative to the radius of the earth, and negligible gravitational effect in the direction of the flight. With these usual assumptions, the equations of motion of the vehicle in its pitch plane can be written as (see Fig. 1)

$$-m(dV/dt) = \tfrac{1}{2}\rho V^2 A C_D \tag{1}$$

$$-mV(d\theta/dt) = \tfrac{1}{2}\rho V^2 A C_L - mg[1 - (V/V_s)^2] \tag{2}$$

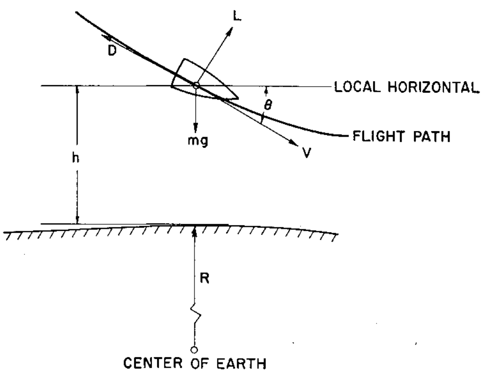

Fig. 1 Geometry of trajectory

The atmosphere can be assumed isothermal, and the density-altitude relation is expressed by

$$\rho = \rho_r e^{-\beta h} \tag{3}$$

Using the relation

$$dh/dt = -V\theta \tag{4}$$

and a transformed variable,

$$X = g\rho_r e^{-\beta h}/(\beta W/A) \tag{5}$$

Eqs. (1) and (2) become

$$(d/dX)(V_s/V)^2 = (C_D/\theta)(V_s/V)^2 \tag{6}$$

$$(V_s/V)^2 = \beta R\, X[\theta(d\theta/dX) + (\tfrac{1}{2})C_L] + 1 \tag{7}$$

Eliminating $(V_s/V)^2$ from these two equations, one obtains

$$\theta^2 \frac{d\theta}{dX} + \theta^2 X \frac{d^2\theta}{dX^2} + \theta X \left(\frac{d\theta}{dX}\right)^2 - C_D\theta X \frac{d\theta}{dX} + \frac{1}{2}\theta\left(C_L + X\frac{dC_L}{dX}\right) = \frac{1}{2} C_L C_D X + \frac{1}{\beta R} C_D \tag{8}$$

This equation of motion applies to a re-entry vehicle having any lift-drag polar and following any lift program. In this investigation, interest will be confined to a parabolic lift-drag polar defined by

$$C_D = C_{D0} + kC_L^2 \tag{9}$$ ‡

The parabolic lift-drag polar closely approximates the aerodynamics of a lifting vehicle of simple geometry, operating at angles of attack below that of $(C_L)_{\max}$. Such a polar lends to mathematical simplicity with no loss of generality. By substitution, Eq. (8) reduces to

$$\theta^2\frac{d\theta}{dX} + \theta^2 X \frac{d^2\theta}{dX^2} + \theta X \left(\frac{d\theta}{dX}^2\right) - \theta X(C_{D0} + kC_L^2)\frac{d\theta}{dX} + \frac{1}{2}\theta\left(C_L + X\frac{dC_L}{dX}\right) = \left(\frac{1}{2} C_L X + \frac{1}{\beta R}\right)(C_{D0} + kC_L^2) \tag{10}$$

To solve this equation, one needs to prescribe a lift program.

The generalized lift program considered in this report is a two-parameter family of variation expressed by

$$C_L = C_{LE} - \zeta X^n \qquad n > 0 \tag{11}$$

It should be noted that the generalized lift program to be studied here can be made to approximate most variable C_L and C_D programs by varying ζ and n. It also can be reduced to the special cases of constant C_L/C_D re-entry, re-entry with constant C_D but variable vertical lift (rolling the vehicle about the velocity vector), and pure ballistic re-entry.

‡ A possible generalization is $C_D = C_{D0} + k_1C_L + k_2C_L^2$.

The generalized lift program and the resultant trajectory are extremely useful in studying the transition phase of re-entry. The transition phase is that portion of the trajectory which bridges the initial plunge and a nominal glide phase. The nominal glide phase can be a constant altitude glide, a constant flight-path angle glide, a constant dynamic pressure glide, or any other mode desired. The generalized lift program permits a smooth transition from the initial plunge and any of the glide phases. The important information derived from the present investigation is not that a smooth transition phase is feasible, but rather what happens to some of the flight variables during transition. It is this information that a preliminary designer needs.

III. General Considerations

The parametric lift program, Eq. (11), contains a term of X^n. It would be very nice if one could solve the equation of motion in the form of a "general solution" with respect to n. That is to say, one only needs to solve Eq. (10) once, and the solution can be applied to all values of n. Unfortunately, this is not possible. (It will be possible if further approximations are made.) In view of this difficulty, solutions were obtained for three values of n, namely, $n = \frac{1}{2}$, 1, 2. (See Ref. 1 for the details.) However, it is worth noting a few general features of the solution to be sought which are independent of the value of n.

Equation (10) is highly nonlinear. A reasonable approach to its solution would be through the use of a series in X of an appropriate form. The boundary conditions at entry always must be satisfied by the solution; they are

$$\theta = \theta_E \tag{12}$$

$d\theta/dX$ must be finite for orbital re-entry, and

$$\theta = \theta_E \qquad X\frac{d\theta}{dX} = \frac{1}{\beta R\theta_E}\left[\left(\frac{V_s}{V_E}\right)^2 - 1\right] \tag{13}$$

for superorbital re-entry.

The first boundary condition for both the orbital and the superorbital cases is obvious. The second boundary conditions are derived from Eq. (7). Close examination of the governing differential equation and its boundary conditions reveals that solutions formed by a simple power series in X would satisfy the boundary condition at the entry point for the orbital case, whereas a series solution of more complicated form would be required to satisfy the boundary condition at the entry point for the case of superorbital re-entry. One source of clue on the selection of the series comes from the studies of Lees, Hartwig, and Cohen[2] and Wang and Ting.[3] Their solutions are written below. For the orbital case[2]

$$\theta = (\theta_E^2 - C_{LE}X)^{1/2} \tag{14}$$

For the superorbital case[3]

$$\theta = \{\theta_E^2 - C_{LE}X - 2Y[1 - (V_s/V_E)^2]\}^{1/2} \tag{15}$$ §

where

$$Y = (1/\beta R)\ln(X/X_E) \tag{16}$$

These approximate solutions were derived under the condition of constant C_L and C_D. In deriving these, additional approximation was made in Refs. 2 and 3 by replacing the velocity in the gravitational term with the entry velocity V_E. Thus, the solution for the case of variable lift will be assumed to have the following form:

$$\theta = \Sigma a_{ij}X^iY^j \qquad i,j \geq 0 \tag{17}$$

The summation sign is understood to be summing on both i and j.

§ At the time of writing, Wang and Ting published their improved solution,[4] of which the present authors were not aware.

Since the representation (17) can be reduced to a simple power series when the coefficients are chosen properly and since the case of orbital re-entry can be considered as a special case of $V_E = V_s$, it becomes obvious that the superorbital and orbital cases can be treated simultaneously using the boundary conditions given by Eq. (13). When these boundary conditions are applied to Eq. (17), one has

$$a_{00} = \theta_E \quad (18)$$

$$a_{01} = (1/\theta_E)[(V_s/V_E)^2 - 1] \quad (19)$$

All the other coefficients will be determined from the governing differential equation. Write Eq. (10) as $F_1 + F_2 = F_3$, where

$$F_1 = \theta^2(d\theta/dX) + \theta^2 X(d^2\theta/dX^2) + \theta X(d\theta/dX)^2$$
$$F_2 = (\tfrac{1}{2})\theta[C_L + X(dC_L/dX)] - \theta X(C_{D0} + k\, C_L^2)(d\theta/dX) \quad (20)$$
$$F_3 = [(\tfrac{1}{2})C_L\, X + (1/\beta R)](C_{D0} + k\, C_L^2)$$

Such a grouping of terms facilitates the computation of the coefficients a_{ij} other than those given by Eqs. (18) and (19) and has no particular advantages otherwise. By substitution, one obtains

$$F_1 = \Sigma\Sigma\Sigma i(i+p)a_{ij}a_{pq}a_{st}X^{i+s+p-1}\, Y^{j+t+q} + (1/\beta R)\Sigma\Sigma\Sigma[i(j+q) + j(i+p)]a_{ij}a_{pq}a_{st} \times X^{i+s+p-1}\, Y^{j+t+q-1} + [1/(\beta R)^2] \times \Sigma\Sigma\Sigma j(j-1+q)a_{ij}a_{pq}\, a_{st}X^{i+s+p-1}\, Y^{j+t+q-2} \quad (21)$$

$$F_2 = (\tfrac{1}{2})[C_{LE} - (n+1)\zeta X^n]\Sigma a_{ij}X^i\, Y^j - (C_{DE} - 2k\zeta C_{LE}\, X^n + k\zeta^2 X^{2n}) \times [\Sigma\Sigma i a_{ij}a_{pq}\, X^{i+p}\, Y^{j+q} + (1/\beta R)\Sigma\Sigma j a_{ij}a_{pq}X^{i+p}\, Y^{j+q-1}] \quad (22)$$

$$F_3 = (C_{DE}/\beta R) + (\tfrac{1}{2})C_{DE}\, C_{LE}X - [k\zeta C_{LE}^2 + (\tfrac{1}{2})\zeta C_{DE}]X^{1+n} + (\tfrac{3}{2})k\zeta^2\, C_{LE}X^{1+2n} - (\tfrac{1}{2})k\zeta^3 X^{1+3n} - (2k\zeta C_{LE}/\beta R)X^n + (k\zeta^2/\beta R)X^{2n} \quad (23)$$

Since F_3 does not contain Y^j, the coefficients a_{0j} and a_{ij}, where $i, j \neq 0$, will be determined from F_1 and F_2. Furthermore, $X^{-1}Y^j(i = p = s = 0)$ can be found only in F_1. Therefore, a_{0j} can be determined from F_1 regardless of the type of the lift program. These coefficients are determined in the following paragraphs.

Setting $i = p = s = 0$, Eq. (21) becomes

$$F_1 = [1/X(\beta R)^2]\Sigma\Sigma\Sigma j(j+q-1)a_{0j}a_{0q}a_{0t}\, Y^{j+t+q-2}$$

Since these are the only terms in the equation of motion which involve $X^{-1}Y^j$, one must have

$$\Sigma\Sigma\Sigma j(i+q-1)a_{0j}a_{0q}\, a_{0t}Y^{j+t+q-2} = 0 \quad (24)$$

This equation determines the coefficients a_{0j} in terms of a_{01}, which has been found in Eq. (19). The following lists the first few of the a_{0j}:

$$a_{02} = -(a_{01}^2/2\theta_E) \qquad a_{03} = (a_{01}^3/2\theta_E^2)$$
$$a_{04} = -(\tfrac{5}{8})(a_{01}^4/\theta_E^3) \quad (25)$$

There is no need to go on any further to identify this particular series. Combining this series with a_{00}, one has

$$a_{00} + \Sigma a_{0j}Y^j = \theta_E[1 + (a_{01}Y/\theta_E) - (a_{01}^2/2\theta_E^2)Y^2 + (a_{01}^3/a\theta_E^2)Y^3 - (\tfrac{5}{8})(a_{01}^4/\theta_E^4)Y^4 + \ldots] = \theta_E\, U_1 \quad (26)$$

and

$$U_1 = [1 + 2(a_{01}/\theta_E)Y]^{1/2} \qquad \text{if} \qquad |2(a_{01}/\theta_E)Y| < 1 \quad (27)$$

For most cases the condition $|2(a_{01}/\theta_E)Y| < 1$ can be satisfied easily.

Next, one collects terms of X^0Y^0, and the resultant equation is

$$(\tfrac{1}{2})C_{LE}\theta_E - (C_{DE}/\beta R)(V_S/V_E)^2 + \theta_E^2 a_{10} + (1/\beta R)(2\theta_E a_{01}a_{10} + 2\theta_E^2 a_{11}) + [1/(\beta R)^2](-\theta_E^2 a_{01}a_{10} + 2\theta_E a_{01}a_{11} + 2\theta_E^2 a_{12}) = 0 \quad (28)$$

This equation contains three coefficients a_{10}, a_{11}, and a_{12}. Before determining these three coefficients, look beyond the terms X^0Y^0 and examine what new coefficients will be introduced when terms of X^0Y^1, X^0Y^2 . . . are collected. It can be seen readily from F_1 that, in addition to the previous coefficients, a_{13} in X^0Y^1, a_{14} in X^0Y^2, a_{15} in X^0Y^3, and so on will be introduced. If a_{10}, a_{11}, and a_{12} can be determined before X^0Y^1 terms are collected, the coefficients a_{1j} can be determined completely. The problem now is to determine a_{10}, a_{11}, and a_{12} by the single Eq. (28). Obviously, three unknowns cannot be determined by one equation, and additional conditions must be introduced.

First of all, one sees that, for orbital re-entry, a double-series solution is not required or, in other words, $a_{ij} = 0$ when $j \neq 0$. This is easily appreciated by comparing Eq. (14) with Eq. (15). Putting this condition in Eq. (28), one finds that for orbital re-entry

$$a_{10} = -(C_{LE}/2\theta_E)\{1 - [2(C_{DE}/C_{LE})/\beta R\theta_E]\} \quad (29)$$

and $a_{01} = a_{11} = a_{12} \ldots = 0$. This indicates that, as V_E becomes slightly different from V_S, the first three terms in Eq. (28) still predominate. On the basis of this physical reasoning, the coefficient a_{10} for cases of V_E slightly different from V_S is determined as

$$a_{10} = -(C_{LE}/2\theta_E) \times \{1 - [2(C_{DE}/C_{LE})/\beta R\theta_E](V_S/V_E)^2\} \quad (29a)$$

and Eq. (29) is a limiting case of Eq. (29a). It is expected that Eq. (29a) is valid even as V_E approaches the escape speed. When a_{10} is so determined, Eq. (28) reduces to

$$(1/\beta R)[2\theta_E a_{01}a_{10} + 2\theta_E^2 a_{11}] + [1/(\beta R)^2] \times [-\theta_E^2\, a_{01}\, a_{10} + 2\theta_E\, a_{01}\, a_{11} + 2\theta_E^2\, a_{12}] = 0 \quad (28a)$$

This equation also contains terms of different orders of magnitude. The quantity βR is large; it is approximately 900 for Earth, 900 for Venus, 200 for Mars, and 3600 for Jupiter. Therefore, $1/\beta R$ is several orders of magnitude higher than $1/(\beta R)^2$. The corresponding terms inside the two brackets are different by no more than one order of magnitude for normal values of θ_E. For example, one can compare $-\theta_E^2\, a_{01}\, a_{10}$ with $2\theta_E a_{01}\, a_{10}$, and the ratio is of the order of θ_E, which is normally about 0.1 for $V_E = 36{,}000$ fps re-entry. Therefore, it is quite reasonable to conclude that the terms involving $1/\beta R$ are of higher order of magnitude than those involving $1/(\beta R)^2$. By this physical reasoning, separate Eq. (28a) as follows:

$$2\theta_E a_{01}a_{10} + 2\theta_E^2 a_{11} = 0 \quad (30)$$
$$-\theta_E^2 a_{01}a_{10} + 2\theta_E a_{01}a_{11} + 2\theta_E^2 a_{12} = 0$$

and the solutions are

$$a_{11} = -(a_{01}/\theta_E)a_{10} \qquad a_{12} = (\tfrac{3}{2})(a_{01}/\theta_E)^2 a_{10} \quad (31)$$

With the coefficients a_{10}, a_{11}, and a_{12} determined, the others of a_{1j} can be determined as just mentioned.

Under one re-entry condition, however, the coefficients a_{1j} take simple forms, and that is when

$$(C_{DE}/\beta R)(V_S/V_E)^2 \ll (\tfrac{1}{2})C_{LE}\theta_E \quad (32)$$

Such a condition can be satisfied for nominal superorbital re-entry; for instance, a vehicle enters the earth atmosphere at $V_E = 36{,}000$ fps, $\theta_E = 6°$, and $C_{LE}/C_{DE} = 1$; condition

(32) reads $0.0012 \ll 0.1$. Assuming condition (32), Eq. (29a) becomes

$$a_{10} = -(C_{LE}/2\theta_E) \quad (33)$$

The coefficients a_{11} and a_{12} remain the same as given in Eq. (31). When the terms of X^0Y^1, X^0Y^2, ... are collected and solved for the unknown coefficients, one obtains

$$a_{13} = -(\tfrac{15}{6})(a_{01}/\theta_E)^3 a_{10} \qquad a_{14} = (\tfrac{35}{8})(a_{01}/\theta_E)^4 a_{10} \quad (34)$$

It is seen readily that $a_{1j}XY^j$ also form a particular series, and it is

$$a_{1j}XY^j = a_{10}X[1 - (a_{01}/\theta_E)Y + (\tfrac{3}{2})(a_{01}/\theta_E)^2Y^2 - (\tfrac{15}{6})(a_{01}/\theta_E)^3Y^3 + \tfrac{35}{8})(a_{01}/\theta_E)^4Y^4 + \ldots] = a_{10}XU_2 \quad (35)$$

and

$$U_2 = [1 + (2a_{01}/\theta_E)Y]^{-1/2} \quad \text{if} \quad |(2a_{01}/\theta_E)Y| < 1 \quad (36)$$

Thus, the basic series for θ becomes

$$\theta = \theta_E U_1 + a_{10}XU_2 + \text{higher order terms determined by specific lift program} \quad (37)$$

The corresponding $(V_E/V)^2$ series is

$$(V_E/V)^2 = 1 + (C_{DE}/\theta_E)X + \text{higher order terms determined by specific lift program} \quad (38)\|$$

Since the coefficients a_{01} and a_{10} are functions of the entry conditions only, the leading terms in both Eqs. (37) and (38) are independent of the lift program. This indicates that lift modulation has little effect on the trajectory at the beginning of the re-entry. If it is permissible to say that the aerodynamics of a re-entry vehicle produces first-order effects on the initial phase of the trajectory (before pull-out), lift and drag modulation only produces higher order effects. Since the present lift program in its parametric form can be used to approximate most lift programs, Eqs. (37) and (38) are valid in most cases. They are especially useful for calculations where first-order approximations are sufficient.

The coefficients a_{2j} will be determined by collecting terms of $X^1Y^0, X^1Y^1, X^1Y^2 \ldots$. When terms of X^1Y^0 are collected, the equation contains coefficients a_{20}, a_{21}, and a_{22}. Again, there are three unknowns to be determined by one equation. These coefficients are determined by exactly the same method as was used to determine a_{10}, a_{11}, and a_{12}. After a_{20}, a_{21}, and a_{22} are determined, the rest of a_{2j} are determined uniquely. When condition (32) is introduced, the $a_{2j}X^2Y^j$ also form a particular series. The higher order coefficients are determined by the same method.

The factor $[1 + (2a_{01}/\theta_E)Y]^{-1/2}$ that appears in each term of the θ series reduces to unity for the case of orbital re-entry, since $a_{01} = 0$ when $V_E = V_s$. Therefore, simple power series representations resulted for both θ and $(V_s/V)^2$ in the case of orbital re-entry, whereas only $(V_E/V)^2$ has simple power series representation in the case of superorbital re-entry.

It is worth noting that, when the lift parameter ζ vanishes (constant C_L and C_D) and the difference between the centrifugal force and the gravitational force is assumed constant (i.e., neglecting terms involving $1/\beta R$), the present solutions for all values of n in the case of $|(2a_{01}/\theta_E)Y| < 1$ reduce to

$$\theta = \theta_E[1 + (2a_{01}/\theta_E)Y]^{1/2} - (C_{LE}/2\theta_E)X[1 + (2a_{01}/\theta_E)Y]^{-1/2} - (C_{LE}^2/8\theta_E^3)X^2[1 + (2a_{01}/\theta_E)Y]^{-3/2} - (C_{LE}^5/16\theta_E^3)X^3[1 + (2a_{01}/\theta_E)Y]^{-5/2} + \ldots \quad (39)$$

|| The first two terms in Eq. (9) of Ref. 4 agree with the present solution when the constant C_1 in Ref. 4 is replaced by -2. The value of C_1 was not given in Ref. 4. By expanding g/V^2 in terms of $\ln V/V_E$, one obtains $C_1 = -2$.

This case corresponds to a constant C_L and C_D program when the velocity is assumed constant and equal to V_E. For this case, Wang and Ting's solution, Eq. (15), becomes valid. It readily is seen that the binomial expansion of that equation yields Eq. (39) for the first four terms.

In the case of orbital re-entry, Eq. (39) reduces to

$$\theta = \theta_E - (C_{LE}/2\theta_E)X - (C_{LE}^2/8\theta_E^3)X^2 - (\tfrac{1}{16})(C_{LE}^3/\theta_E^5)X^3 + \ldots \quad (40)$$

For this case, Lees, Hartwig, and Cohen[2] obtained a simple solution as given by Eq. (14). Expanding Eq. (14) by the binomial theorem, one gets Eq. (40) for the first four terms. This formal reduction serves as a check against the present series solutions. Furthermore, the present solutions including the $1/\beta R$ terms can be used to estimate the error introduced by assuming the difference between the centrifugal force and the gravitational force constant. To estimate this error, one can replace the series solution (see Ref. 1 for the details) by

$$\theta = \theta_E \left\{1 - \left(\frac{C_{LE}}{\theta_E^2}\right)\left[1 - \frac{2(C_{DE}/C_{LE})}{\beta R\theta_E}\left(\frac{V_s}{V_E}\right)^2\right]X + \left(\frac{2a_{01}}{\theta_E}\right)Y\right\}^{1/2} \quad (41)$$

[When $C_1 = -2$ is substituted in Eq. (10) of Ref. 4 and $\rho_E = 0$ is assumed, that equation reduces to the present solution with the exception of a small term that is proportional to X^2. If the X^2 term is included in the present approximation, the two solutions approach each other.] This equation is valid for the constant lift program ($\zeta = 0$). The corresponding expression of θ for the orbital case can be obtained by letting $V_E = V_s$ and $a_{01} = 0$ in Eq. (41).

When the lift parameter ζ is only slightly greater than zero, the following approximation is valid:

$$\theta = \theta_E \left\{1 - \left(\frac{C_{LE}}{\theta_E^2}\right)\left[1 - \frac{2(C_{DE}/C_{LE})}{\beta R\theta_E}\left(\frac{V_s}{V_E}\right)^2\right]X + \left(\frac{2a_{01}}{\theta_E}\right)Y + \left[\frac{\zeta}{(n+1)\,\theta_E^2}\right] \times \left[1 - \frac{4kC_{LE}}{(n+1)\beta R\theta_E}\left(\frac{V_s}{V_E}\right)^2\right]X^{1+n}\right\}^{1/2} \quad (42)$$

The expression of θ corresponding to Eq. (42) for the orbital case is readily obtainable.

These equations, Eqs. (41) and (42), will be used in determining the peak deceleration and the altitude where it occurs for $\zeta = 0$ and for small values of ζ. They also will be used in studying the transition phase.

IV. Deceleration Load

In this section, the peak deceleration load with lift modulation is studied. The heating problem is not considered here, because the heat input to a vehicle is strongly configuration-dependent, and a general parametric study of heating cannot be carried out with a sufficient degree of realism. On the other hand, the deceleration load can be studied parametrically as long as a lift-drag polar is specified.

The deceleration load due to aerodynamic forces is expressed by

$$G = (\tfrac{1}{2})\rho V^2 AC_R/W = (\beta R/2)XC_R(V/V_s)^2 \quad (43)$$

The peak deceleration occurs where $dG/dX = 0$, i.e.,

$$C_DX = \theta[1 + (X/C_R)(dC_R/dX)] \quad (44)$$

The total force coefficient C_R decreases as the vehicle descends according to the present lift program, and θ is positive at peak G. Thus one finds that, at peak G, the following condition is true:

$$-1 < (X/C_R)(dC_R/dX) < 0 \tag{45}$$

The two limiting cases

$$\left(\frac{X}{C_R}\right)\left(\frac{dC_R}{dX}\right) = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$$

are of interest and also simplify the calculations of X^* and G^*. The first limit, i.e., $(X/C_R)(dC_R/dX) = 0$, corresponds to the case of constant C_L and C_D. The second limit, i.e., $(X/C_R)(dC_R/dX) = -1$, corresponds to large values of ζ. When ζ is large, C_R decreases rapidly with X, and the peak deceleration would occur at a point where the velocity has not yet decayed appreciably. In this case, the peak deceleration is a result of the increasing density and decreasing total force coefficient. In the first limiting case, the peak deceleration results from the increasing density and decreasing velocity.

A. First Limiting Case: $(X/C_R)(dC_R/dX) = 0$

For this case, Eq. (44) is reduced to

$$C_{DE}X_0^* = \theta_0^* \tag{46}$$

where X_0^* denotes the X value where peak G occurs for the case $\zeta = 0$, and θ_0^* is the corresponding flight-path angle. The flight-path angle θ for $\zeta = 0$ has been given in Eq. (41). In that equation, the term containing $Y = (1/\beta R) \ln(X/X_E)$ makes the computation of X_0^* cumbersome. However, it can be simplified by noting that the critical altitude is usually in the range of 170,000 to 250,000 ft, where (X/X_E) is much greater than unity. One can approximate Y by

$$Y = K_0 + K_1X + K_2X^2 \tag{47}$$

where K_0, K_1, and K_2 are constants. In the altitude range of interest, $K_0 = 0.0073$, $K_1 = 7.26 \times 10^{-7} X_E^{-1}$, $K_2 = -3.85 \times 10^{-11} X_E^{-2}$ provide a good approximation. By this approximation, Eqs. (41, 46, and 47) give

$$\theta_E^2 + 2\theta_E a_{01}K_0 + 2\theta_E a_{01}K_1X_0^* + 2\theta_E a_{01}K_2X_0^{*2} - C_{LE}\left[1 - \frac{2(C_{DE}/C_{LE})}{\beta R\theta_E}\left(\frac{V_s}{V_E}\right)^2\right]X_0^* = C_{DE}X_0^{*2} \tag{48}$$

Solving this quadratic equation for X_0^* for the case $[2(C_{DE}/C_{LE})/\beta R\theta_E]\cdot(V_s/V_E)^2 \ll 1$, one has

$$\frac{M\, C_{DE}X_0^*(C_{DE}/C_{LE})}{P} = \left[1 + \frac{4(C_{DE}/C_{LE})^2(\theta_E^2 + 2\theta_E a_{01}K_0)M}{P^2}\right]^{1/2} - 1 \tag{49}$$

where

$$M = 1 - (2\theta_E a_{01}K_2/C_{DE}^2)$$
$$P = 1 - (2\theta_E a_{01}K_1/C_{LE})$$

Note here that the second term under the square root is usually much smaller than unity. For such cases, Eq. (49) can be reduced further to

$$X_0^* = (1/C_{LE})[\theta_E^2 + 2\theta_E a_{01}K_0]/[1 - (2\theta_E a_{01}K_1/C_{LE})] \tag{50}$$

For orbital case ($a_{01} = 0$), Eq. (50) reduces to

$$X_0^* = \theta_E^2/C_{LE} \tag{51}$$

Comparing Eqs. (50) and (51), it is seen clearly that, for a given re-entry condition, the peak deceleration of the superorbital case occurs at a higher altitude than that of the orbital case. Furthermore, X_0^* in the superorbital case depends on (K_1/C_{LE}), which is directly proportional to $(W/C_{LE}A)$, whereas that in the orbital case is independent of $(W/C_{LE}A)$.

Since, according to Eq. (38), the velocity may be approximated by

$$(V_E/V)^2 = 1 + (C_{DE}/\theta_E)X \tag{52}$$

the peak deceleration therefore is obtained from Eq. (43) as

$$G_0^* = (\beta V_E^2\theta_E/2g)[1 + (C_{LE}/C_{DE})^2]^{1/2} \times [C_{DE}X_0^*/(\theta_E + C_{DE}X_0^*)] \tag{53}$$

where X_0^* should be determined either from Eq. (49) or (50).

For small values of ζ, the θ solution has been approximated by Eq. (42). It can be reduced further to

$$\theta = \{\theta_E^2 - C_{LE}X + 2a_{01}\theta_E Y + [\zeta/(n+1)]X^{1+n}\}^{1/2} \tag{54}$$

for cases of

$$[2(C_{DE}/C_{LE})/\beta R\theta_E] \ll 1 \text{ and } [4kC_{LE}/\beta R(n+1)\theta_E] \ll 1$$

Note that these inequalities usually are satisfied by practical cases of re-entry into earth atmosphere.

The drag coefficient is given by

$$C_D = C_{DE} - 2k\zeta C_{LE}X^n + k\zeta^2X^{2n} \tag{55}$$

Substituting Eqs. (54) and (55) into Eq. (44) and differentiating the resulting equation with respect to ζ and then letting $\zeta = 0$, $X^* = X_0^*$, and $Y^* = Y_0^*$, one obtains the initial slope given below:

$$[2C_{DE}X_0^*(C_{DE}/C_{LE})] + \{1 - [2\theta_E a_{01}(K_1 + K_2X_0^*)/C_{LE}]\} \times [(d/dZ)(X^*/X_0^*)]_0 = [(C_{DE}X_0^*)^n/(n+1)] + 4(C_{DE}/C_{LE})(C_{DE}X_0^*)^{1+n}(kC_{LE}^2/C_{DE}) - \{n(1 + 2kC_{DE})/2[1 + (C_{DE}/C_{LE})^2]\} \tag{56}$$

where

$$Z = (\zeta/C_{LE})(1/C_{DE})^n \tag{57}$$

The second term on the right-hand side of Eq. (56) can be neglected because of the high power in $C_{DE}X_0^*$. Therefore,

$$\left[\frac{d}{dZ}\left(\frac{X^*}{X_0^*}\right)\right]_0 = \frac{[(C_{DE}/C_{LE})^n/2^n(n+1)]\;[2C_{DE}X_0^*(C_{DE}/C_{LE})^n]}{1 - [2a_{01}\theta_E(K_1 + K_2X_0^*)/C_{LE}] + 2C_{DE}X_0^*(C_{DE}/C_{LE})} \tag{58}$$

where X_0^* can be taken from Eq. (49) or (50). Equation (58) shows that the initial slope of X^*/X_0^* with respect to Z(i.e., ζ) is positive for superorbital re-entry but decreasing with increasing entry velocity. When the entry velocity is suborbital, the initial slope of X^*/X_0^* remains to be positive but increases with decreasing velocity. It must be remembered, however, that only suborbital re-entry with an entry speed close to orbital is being mentioned, because very low entry speeds usually are accompanied by high entry angles that violate one basic assumption of this study.

The next quantity to be determined is the initial slope of (G^*/G_0^*). By Eq. (43), one can write

$$(G^*/G_0^*) = (X^*/X_0^*)(C_R^*/C_{RE})[(1 + b_1X_0^*)/(1 + b_1X^*)] \tag{59}$$

where $(V_E/V)^2 = 1 + b_1X$, with $b_1 = (C_{DE}/\theta_E)$, has been used. This approximation is acceptable because the velocity at peak G for small ζ should be close to that for $\zeta = 0$, since the value of V^* is not too much different from V_E in any case. Differentiating Eq. (59) with respect to Z and then letting $Z \to 0$, $X^* \to X_0^*$, and $C_R^* \to C_{RE}$, one obtains

$$\frac{[(d/dZ)(G^*/G_0^*)]_0}{(C_{DE}\,X_0^*)^n} = \frac{\theta_E/(\theta_E + C_{DE}X_0^*)}{(n+1)[2C_{DE}X_0^*(C_{DE}/C_{LE}) + 1]} - \frac{1 + 2kC_{DE}}{1 + (C_{DE}/C_{LE})^2} \tag{60}$$

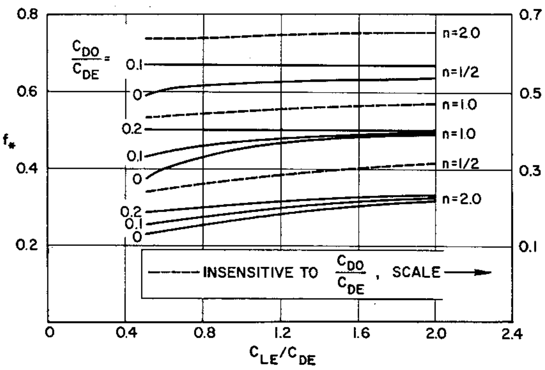

Fig. 2 C_R^*/C_{RE} **and** f_* **as a function of** C_{LE}/C_{DE} **(solid curves,** f_***; dashed curves,** C_R^*/C_{RE}**)**

In Eq. (60), the first term on the right-hand side is always less than unity, whereas the second term can be less than, equal to, or greater than unity because

$$\frac{1 + 2kC_{DE}}{1 + (C_{DE}/C_{LE})^2} = 1 + \frac{(C_{DE}/C_{LE})^2[1 - (2C_{D0}/C_{DE})]}{1 + (C_{DE}/C_{LE})^2} \quad (61)$$

For cases when $C_{DE} \geq 2C_{D0}$, i.e., if the initial angle of attack is equal to or greater than that of $(C_L/C_D)_{\max}$, one must have

$$(1 + 2kC_{DE})/[1 + (C_{DE}/C_{LE})^2] \geq 1 \quad (62)$$

and, consequently,

$$[(d/dZ)(G^*/G_0^*)]_0 < 0 \quad (63)$$

For cases when $C_{DE} < 2C_{D0}$, the initial slope of G^*/G_0^* still may be negative depending on the other entry conditions. In actual practice, the initial angle of attack can be on either side of that of $(C_L/C_D)_{\max}$, depending on the type of maneuvering desired. It must be remembered that a small amount of lift modulation may not reduce the peak deceleration if the initial angle of attack is such that $C_{DE} < 2C_{D0}$. These conclusions are applicable equally to orbital and superorbital re-entries.

B. Second Limiting Case: $(X/C_R)(dC_R/dX) = -1$

The full expression of the total force coefficient is given by

$$C_R^2 = C_{RE}^2 - 2(1 + 2kC_{DE})C_{LE}\zeta X^n + (1 + 2kC_{DE} + 4k^2C_{LE}^2)(\zeta X^n)^2 - 4k^2C_{LE}(\zeta X^n)^3 + k^2(\zeta X^n)^4 \quad (64)$$

where $C_{RE}^2 = C_{LE}^2 + C_{DE}^2$. The condition $(X/C_R)(dC_R/dX) = -1$ therefore is reduced to

$$1 + (C_{DE}/C_{LE})^2 - (2 + n)(1 + 2kC_{DE})(\zeta X^{*n}/C_{LE}) + (1 + n)(1 + 2kC_{DE} + 4k^2C_{LE}^2)(\zeta X^{*n}/C_{LE})^2 - 2(2 + 3n)k^2C_{LE}^2(\zeta X^{*n}/C_{LE}) + (1 + 2n)k^2C_{LE}^2(\zeta X^{*n}/C_{LE})^4 = 0 \quad (65)$$

According to the lift-drag polar, one has

$$kC_{DE} = (C_{DE}/C_{LE})^2[1 - (C_{D0}/C_{DE})] \quad (66)$$

$$k^2C_{LE}^2 = (C_{DE}/C_{LE})^2[1 - (C_{D0}/C_{DE})]^2 \quad (67)$$

Equation (65) can be solved for $(\zeta X^{*n}/C_{LE})$ in terms of C_{LE}/C_{DE}, C_{D0}/C_{DE}, and n. Denoting $(\zeta X^{*n}/C_{LE})$ by f^*, one has

$$C_{DE}X^* = (f^*)^{1/n}[1/(\zeta/C_{LE})(1/C_{DE})^n]^{1/n} = (f^*/Z)^{1/n} \quad (68)$$

Therefore, for large values of ζ, and when Eq. (50) is used for X_0^*, one has

$$(X^*/X_0^*) = (f^*/Z)^{1/n}(C_{LE}/C_{DE}) \times [1 - (2\theta_E a_{01}K_1/C_{LE})]/[\theta_E^2 + 2\theta_E a_{01}K_0] \quad (69)$$

The ratio of G^*/G_0^* thus is given by

$$(G^*/G_0^*) = (X^*/X_0^*)(C_R^*/C_{RE}) \times [(\theta_E + C_{DE}X_0^*)/(\theta_E + C_{DE}X^*)] \quad (70)$$

where f^* and (C_R^*/C_{RE}) are plotted in Fig. 2. It is clear that both X^*/X_0^* and G^*/G_0^* decrease as Z (and hence ζ) increases for large values of ζ.

It should be pointed out that Eq. (65) is quartic, and there is no guarantee that it has a real root. Under this circumstance, the deceleration has no peak for the second limiting case.

C. Matching of Small ζ and Large ζ Solutions

Now that one has computed X^*/X_0^* and G^*/G_0^* for large values of ζ and the initial slopes of these two quantities, the remaining of the work is to match the two branches of the solutions. No simple analytic method can be found to do the matching, and help must be obtained from other sources. For this purpose, the authors turn to an analog computer program that also is used for checking the analytical solutions derived.

The matching of the two branches of solutions is demonstrated by two sample cases. The first sample case corresponds to orbital re-entry of $n = 1$, $\theta_E = 6°$, $(C_{LE}/C_{DE}) = 1$, and $(C_{D0}/C_{DE}) \ll 1$; and the second sample case corresponds to superorbital re-entry of $n = 2$, $\theta_E = 9°$, $(C_{LE}/C_{DE}) = 1$, and $(C_{D0}/C_{DE}) \ll 1$. Both cases have been computed by the foregoing equations and by the analog computer program. Results are shown in Fig. 3 for the orbital case and in Fig. 4 for the superorbital case. The analog results show extremely good agreement with this analysis for both small and large ζ. The analog results also indicate a fairly large radius of convergence for the second limiting case of large ζ, which makes the matching straightforward. The characteristics of the matching of these two cases are typical of all entry conditions.

V. Transition Phase

In this section, the transition phase that bridges the initial plunge and a nominal glide phase will be studied. The nominal glide phases considered are 1) constant flight-path angle, 2) constant altitude, and 3) constant dynamic pressure.

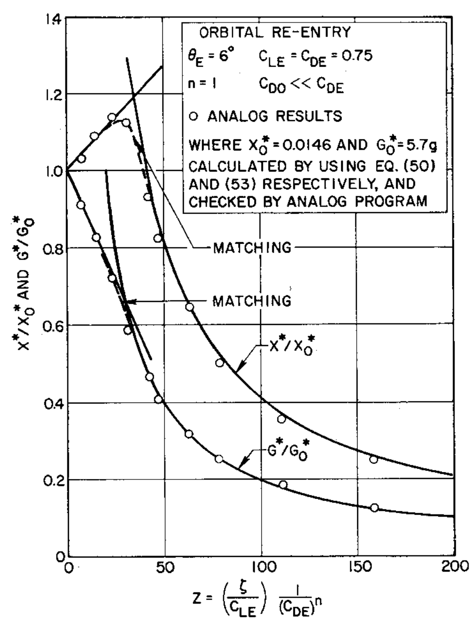

Fig. 3 Variation of altitude parameter (X^*/X_0^*) and peak deceleration ratio (G^*/G_0^*) as a function of lift parameter Z; example, orbital re-entry

The transition phase initiates at the entry point and joins the glide phase at some altitude where the flight and aerodynamic variables of the two phases are matched. When V, θ, C_L, and C_D are matched, the first derivatives or V and θ are matched automatically. This can be seen readily from Eqs. (6) and (7). This results in what might be called a continuous control, although the first derivatives of C_L and C_D will not be matched. Such a control concept is actually the common practice in airplane flight.

The matching of the trajectory is carried out only to the first-order approximation in ζ. The solutions for θ and $(V_E/V)^2$ used in the transition phase are taken to be

$$\theta = [\theta_E^2 - C_{LE}X + 2a_{01}\theta_E Y + (\zeta/n + 1)X^{1+n}]^{1/2} \quad (71)$$

$$(V_E/V)^2 = 1 + (C_{DE}X/\theta_E)\{1 + (C_{LE}X/4\theta_E^2) \times [1 + 2\theta_E(C_{DE}/C_{LE})] - [2kC_{LE}\zeta X^n/(n + 1)C_{DE}]\} \quad (72)$$

where Eq. (72) is obtainable readily from equations given in Appendix A of Ref. 1.

These solutions satisfy the conditions at one boundary, namely, the point of entry. The task here is to determine the lift program (ζ and n) which results in a trajectory that satisfies the conditions at another boundary, namely, the end of the transition phase. This additional boundary condition will be derived from each specific glide phase.

After the required lift program is obtained, it then can be used in conjunction with numerical integration of the equations of motion to obtain the other trajectory parameters needed for the determination of re-entry heating, range during re-entry, etc.

A. Transition into a Constant Flight-Path Angle Glide

In this case, the vehicle is required to experience just the right amount of lift modulation to turn into a constant flight-path angle glide. The constant flight-path angle θ_1 is assumed to be smaller than the entry angle θ_E. A constant flight-path angle glide is defined by

$$\theta = \theta_1 \quad (73)$$

and

$$(\tfrac{1}{2})\rho V^2 A C_L = mg \cos\theta_1[1 - (V/V_s)^2] \quad (74)$$

Equation (74) is the same as $(d\theta/dX) = 0$. The reason for using Eq. (74) is twofold. First, Eqs. (73) and (74) so constructed are also the conditions of a constant altitude flight when θ_1 is set to zero; thus the solutions obtained here are applicable also for transition into a constant altitude glide. Second, one must use the approximated trajectory solutions

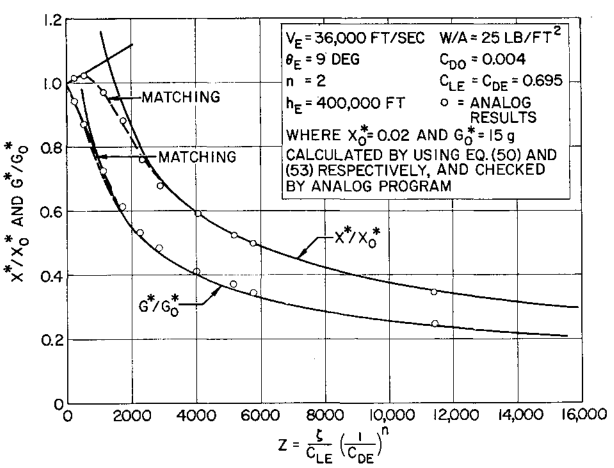

Fig. 4 Variation of the altitude parameter (X^*/X_0^*) and peak deceleration ratio (G^*/G_0^*) as a function of the lift parameter Z; example, superorbital re-entry

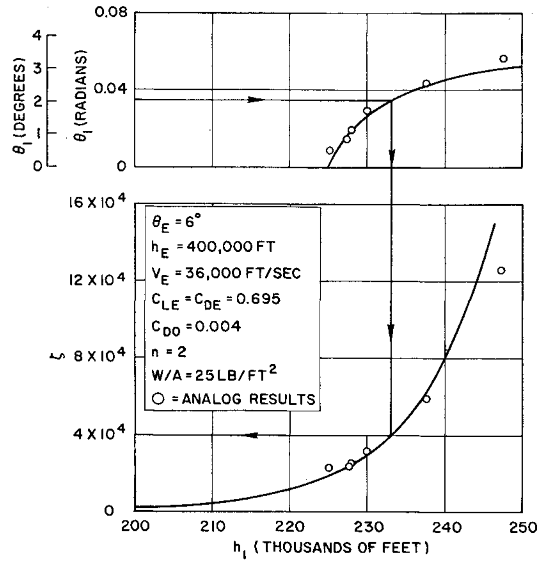

Fig. 5 Transition into constant flight-path angle glide; superorbital re-entry

obtained in this paper for the matching of the transition phase with the glide phase; consequently, use of the approximated solution of θ to obtain its derivative should be avoided in order to maintain a reasonable accuracy. Equation (74) also can be written as

$$(\beta RX/2\cos\theta_1)C_L = [(V_s/V)^2 - 1] \quad (75)$$

To match the flight-path angle θ, one has by Eqs. (71) and (73)

$$\theta_1^2 = \theta_E^2 - C_{LE}X_1 + 2a_{01}\theta_E Y_1 + [\zeta/(n + 1)]X_1^{n+1} \quad (76)$$

where $Y_1 = K_0 + K_1X_1 + K_2X_1^2$. Furthermore, Eq. (75) becomes

$$(\beta RX_1/2\cos\theta_1)(C_{LE} - \zeta X_1^n) + 1 = (V_s/V_E)^2\{1 + (C_{DE}X_1/4\theta_E^2) \times (1 + (C_{LE}X_1/4\theta_E^2)[1 + 2\theta_E(C_{DE}/C_{LE})] - [2kC_{LE}/(n + 1)C_{DE}]\zeta X_1^n)\} \quad (77)$$

Equation (77) can be solved for ζX_1^n, and, with a slight sacrifice in accuracy ($\pm 3\%$ in $\zeta X_1^n/C_{LE}$), one obtains

$$\zeta X_1^n/C_{LE} = 0.97 + [1 - (V_s/V_E)^2] \times [2\cos\theta_1/\beta R(C_{LE}X_1)] \quad (78)$$

As far as ζ and n are concerned, an inaccuracy of 3% is tolerable. The fact that one is able to make this approximation indicates that this transition phase is insensitive to k, which is the characteristic of the lift-drag polar. Consequently, Eq. (78) can be applied to any lift-drag polar, parabolic or otherwise. The parameter n is determined by

$$\frac{1}{n + 1} = \frac{[1 - (2\theta_E a_{01}K_1/C_{LE})] + [\theta_1^2 - (\theta_E^2 + 2a_{01}\theta_E Y_1)]/C_{LE}X_1}{0.97 + [1 - (V_s/V_E)^2](2\cos\theta_1/\beta R C_{LE}X_1)} \quad (79)$$

The limit on X_1 is

$$C_{LE}X_1 > (\theta_E^2 + 2a_{01}\theta_E Y_1 - \theta_1^2)/[1 - (2a_{01}\theta_E K_1/C_{LE})] \quad (80)$$

because the expression (79) is necessarily positive.

A sample calculation is given in Fig. 5. The analog results agree very well with the prediction for small ζ. For large ζ some deviation is observed, and this is to be expected because Eqs. (78) and (79) are derived for small values of ζ. In

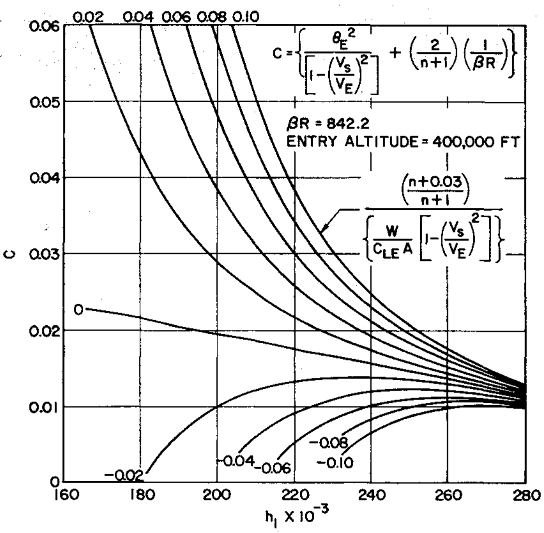

Fig. 6 Transition into constant altitude glide; superorbital re-entry

practice, the constant flight-path angle is always on the low side, and the present analysis appears to be quite adequate.

For the case of orbital re-entry, Eq. (78) further reduces to

$$X_1 = (0.97C_{LE}/\zeta)^{1/n} \tag{81}$$

Substituting Eq. (81) into (76), one obtains

$$\theta_1^2 = \theta_E^2 - [(n + 0.03)/(n + 1)] \times (0.97C_{LE}/\zeta)^{1/n}C_{LE} \tag{82}$$

Note that the values of X_1 resulting from Eq. (81) automatically satisfy the inequality

$$X_1 > (\theta_E^2 - \theta_1^2)/C_{LE} \tag{83}$$

This condition corresponds to the fact that a given flight-path angle θ_1 always occurs at a lower altitude with the present lift program than with a constant C_L and C_D program.

B. Transition into a Constant Altitude Glide

The constant altitude glide is a special case of the constant flight-path angle glide when $\theta_1 = 0$. The results of the last section apply. Because of the practical interest of a constant altitude glide, it is worked out in some detail.

The equation for the parameter n, Eq. (79), may be rewritten as

$$\frac{\theta_E^2}{1 - (V_s/V_E)^2} + \left[\frac{2}{\beta R(n + 1)}\right] = \left[\frac{(n + 0.03)}{(n + 1)}\right] \times \left[\frac{C_{LE}X_1}{1 - (V_s/V_E)^2}\right] + 2Y_1 \tag{84}$$

For a given set of values of n, θ_E, and V_E, the left-hand side of Eq. (84) is determined. Denote this quantity by C. Equation (84) therefore is reduced to

$$C = [(n + 0.03)/(n + 1)] \times \{C_{LE}X_1/[1 - (V_s/V_E)^2]\} + 2Y_1 \tag{85}$$

Equation (85) is plotted in Fig. 6 in terms of the altitude h_1. For example, given $\theta_E = 6°$, $V_E = 36{,}000$ fps, $C_{LE} = C_{DE} = 0.695$, $W/A = 25$ psf, $C_{D0} \ll C_{DE}$, $k = 1.44$, and choosing $n = 2$, one computes $C = 0.0228$ and

$$[(n + 0.03)/(n + 1)]\{(W/C_{LE}A) \times [1 - (V_s/V_E)^2]\}^{-1} = 3.76 \times 10^{-2}$$

One finds from Fig. 6 that $h_1 = 225{,}000$ ft. The corresponding value of ζ as computed from Eq. (92) is $\zeta = 2.12 \times 10^4$. The lift coefficient $C_{L_1} = -0.16$ at the end of transition.

The transition trajectory has been computed on the analog computer. The results are presented in Fig. 7.

In the case of orbital re-entry, Eq. (84) reduces to

$$(n + 0.03)/(n + 1) = \theta_E^2/C_{LE}X_1 \tag{86}$$

This equation gives the altitude function X_1 in terms of n, θ_E, and $W/C_{LE}A$. For example, if the re-entry angle is 4°, $W/C_{LE}A = 50$ psf, and $n = 1$ is chosen, one has $C_{LE}X_1 = 0.0095$ from Eq. (86) and $X_1W/A = 0.475$. The corresponding value of h_1 is found to be $h_1 = 198{,}000$ ft, and the value of ζ can be computed from Eq. (81) and C_{L_1} from

$$C_{L_1} = C_{LE} - \zeta X_1^n \tag{87}$$

In both the superorbital and orbital cases, the larger the value of n, the higher the altitude h_1 becomes.

C. Transition into a Constant Dynamic Pressure Glide

A constant dynamic pressure glide is defined by

$$(\tfrac{1}{2})\rho V^2 = q_0 = \text{const} \tag{88}$$

In terms of the normalized variables, this equation becomes

$$(\beta R/2q_0)(W/A)X = (V_s/V_E)^2(V_E/V)^2 \tag{89}$$

Substituting this condition into Eqs. (6) and (7), the equations of motion are reduced to

$$\theta = C_DX \tag{90}$$

$$(V_E/V_s)^2(W/2q_0A) - (1/\beta RX) = C_DX[C_D + X(dC_D/dX)] + (\tfrac{1}{2})C_L \tag{91}$$

Equations (90) and (91) must be satisfied along a constant dynamic pressure glide and, therefore, at the point of transition. In order to have a smooth transition, the trajectory of the transition phase must satisfy conditions (90) and (91) at its point of termination, say X_2. Therefore, the transition trajectory must be such that at X_2

$$\theta_2 = C_{D_2}X_2 \tag{92}$$

and

$$(V_E/V_s)^2(W/2q_0A) - (1/\beta RX_2) = C_{D_2}X_2[C_{D_2} + X_2(dC_D/dX)_2] + (\tfrac{1}{2})C_{L_2} \tag{93}$$

When the expressions for θ, C_D, and C_L from Sec. III are substituted into these two equations, they permit the determination of any two among the four parameters ζ, n, q_0, and X_2.

For purpose of illustration, consider the case of roll control. In this case, the vehicle is rolled about the freestream velocity vector, thus changing the vertical lift but keeping the angle of attack constant. It is seen readily that a roll control program in which the total force coefficient remains constant corresponds to a special case of $k = 0$ of the general program.

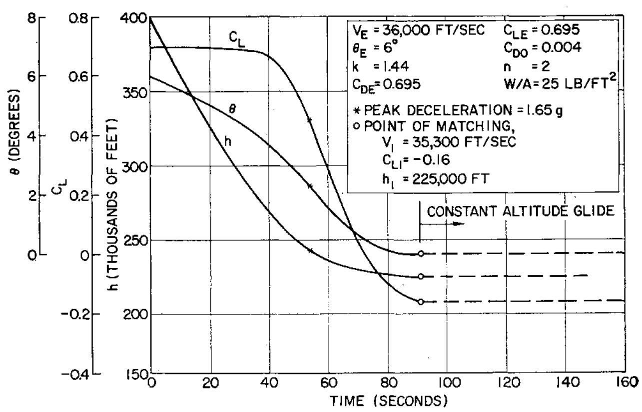

Fig. 7 History of a sample transition trajectory, transition into constant altitude; superorbital re-entry

Consequently, Eqs. (92) and (93) may be reduced to

$$\zeta X_2^n/C_{LE} = 1 - (V_E/V_s)^2(W/2q_0AC_{LE}) + (2/\beta RC_{LE}X_2) + 2C_{LE}X_2(C_{DE}/C_{LE})^2 \quad (94)$$

$$(n+1) = (\zeta X_2^n/C_{LE})C_{LE}X_2[C_{DE}^2X_2^2 - (\theta_E^2 - C_{LE}X_2 + 2a_{01}\theta_E Y_2)]^{-1} \quad (95)$$

Parametrically, Eqs. (94) and (95) may be used to determine any two among the four parameters ζ, n, X_2, and q_0 when the other two are prescribed along with a given set of conditions at the point of re-entry.

An example of the roll control into a constant dynamic pressure glide from orbital velocity is given in Fig. 8. The re-entry conditions are $\theta_E = 3°$, $V_E = V_s = 25{,}750$ fps, $C_{LE} = C_{DE} = 0.75$, $C_{D0} = 0.025$, $W/A = 25$ psf. The transition altitude is 222,000 ft and $\zeta = 103$ as computed from Eqs. (94) and (95) for $n = 1$ and $q_0 = 82$ psf. The trajectory, as computed by Eq. (71), is given in the h-θ plot, which clearly demonstrates the characteristics of the transition phase.

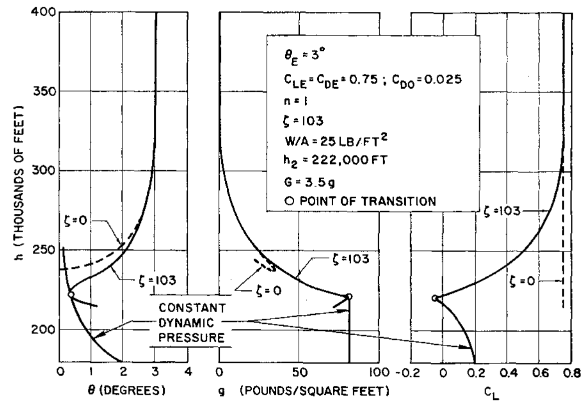

Fig. 8 Example of transition into constant dynamic pressure glide by roll control; orbital re-entry

VI. Concluding Remarks

The present series approach to the trajectory solution has been proved successful for determining the initial portion of a lifting re-entry trajectory. It not only made the determination of the solutions manageable, but it also brought out the fact that lift and drag modulation exhibits only second- or higher-order effects on the trajectory. Although only the first few terms of the series have been determined, they are sufficient for studying the transition phase and for calculating the peak deceleration. The lift modulation program considered here is a general one; it can be used to approximate most practical lift programs. The general trajectory solutions include the special cases of constant C_L/C_D re-entry, roll control, and ballistic re-entry.

The peak deceleration with lift modulation has been calculated for two limiting cases: 1) for $\zeta = 0$ and small ζ, and 2) for large ζ. The case of $\zeta = 0$ corresponds to constant C_L and C_D flight. For small ζ, the initial slopes of G^* and X^* have been obtained. For large ζ, the peak deceleration is primarily a result of the increase of the atmospheric density and the decrease of the total force coefficient. The results indicate that the peak deceleration always decreases with increasing amount of lift reduction if the angle of attack at the point of entry is higher than that of $(C_L/C_D)_{max}$. The altitude where peak deceleration occurs, however, first decreases with increasing lift reduction and then increases. These predictions have been shown to agree very well with the results of an analog program.

One of the applications of lift modulation is to achieve a smooth transition into a nominal glide phase from the initial plunge. Transitions into three nominal glide phases (constant θ, constant h, and constant q) have been studied, among which the transition into a constant altitude glide has been worked out in detail. Simple formulas and graphs have been provided. A smooth transition into constant deceleration glide is also of interest but is not studied here because of mathematical difficulty. If roll control is used, however, the constant G glide becomes the constant q glide, and the results of the latter can be used.

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AIAA TORPEDO PROPULSION CONFERENCE

(Confidential)

Under Co-Sponsorship of the U. S. Naval Underwater Ordnance Station

U. S. NAVAL UNDERWATER ORDNANCE STATION JULY 23–26, 1963 NEWPORT, RHODE ISLAND

With the active cooperation of the Bureau of Naval Weapons, the Naval Underwater Ordnance Station, and the U. S. Naval Ordnance Test Station, Pasadena, the AIAA Underwater Propulsion Committee is organizing its first conference on torpedo propulsion, preparing a *Confidential* Compendium, and considering preparation of an unclassified AIAA Progress Series volume.

The purpose of the conference and of the publications is the same: to provide information to a national audience on the problems, progress, goals, and innovations of torpedo-propulsion activities in the United States. The information will be stimulating and useful to scientists, engineers, and administrators from government and industry who are knowledgeable and who are contributing in this area. In addition, it will prove valuable to those who wish to become familiar with underwater activities.

Session topics are as follows:

Modern Torpedoes—Mission, Development, and Power Plants
Future Torpedo Concepts—Missions and Power Plants
Future Torpedoes—Their Energy Sources
Future Torpedoes—Their Prime Movers
Power-Plant Noise Reduction
Up-Dating Recent Papers
Future of Torpedo Propulsion (Panel)